This review follows closely the apt summary provided by the authors in the first section of the paper.

Let $\mathsf{Ar}\left(\mathbb{Q}^2\right)$ be the space of compatible total orders on $\mathbb{Q}^2$ satisfying the Archimedean property (any group having such an order is embeddable in $(\mathbb{R},+,<)$ and hence Abelian). In Section 2, it is proved that there is a countable-to-one Borel homomorphism from $\langle \mathbb{R}\setminus\mathbb{Q}, E_{\mathrm{GL}}\rangle$ to $\langle \mathsf{Ar}\left(\mathbb{Q}^2\right), \cong\rangle$, where E_{GL} is the orbit equivalence relation on $\mathbb{R}\setminus\mathbb{Q}$ corresponding to the action of $\mathrm{GL}_2(\mathbb{Z})$ by fractional linear transformations. This shows that $\cong_{\mathsf{Ar}(\mathbb{Q}^2)}$ is not smooth.

In Section 3 it is proved that the first Friedman-Stanley jump $=^+_{\mathbb{R}}$ of the equality on reals is reducible to the isomorphism relation on the space ArGp of countable Archimedean groups. More precisely, Section 4 relates $\cong_{\mathsf{ArGp}}$ to the refined hierarchy $\cong^*_{n,k}$ from [Hjorth, G.; Kechris, A.S.; Louveau, A.; Borel equivalence relations induced by actions of the symmetric group. Ann. Pure Appl. Logic **92** (1998), no. 1, 63–112. MR1624736] as follows:

- $\cong_{\mathsf{ArGp}} \leq_B \cong^*_{3,1} <_B =^{++}_{\mathbb{R}}$. Indeed, an Borel assignment of invariants witnessing the first reduction is essentially given by

$$G \mapsto A_G := \left\{\{\tfrac{g}{r} \mid g\in G\} \mid r \in G\setminus\{0\}\right\},$$

- $\cong_{\mathsf{ArGp}} \not\leq_B \cong^*_{3,0}$, and hence ArGp does not admit a Borel assignment of countable sets of reals as complete invariants.

The second item above is an application of the method by [Shani, A.; Borel reducibility and symmetric models. Trans. Amer. Math. Soc. **374** (2021), no. 1, 453–485. MR4188189] in which non-reducibility results are obtained by considering *generic invariants*, i.e., $\tilde{A}_G$ for G generic over some ground model (and $z \mapsto \tilde{A}_z$ is a Borel assignment of complete invariants). For this particular result, the ground is the basic Cohen model of $\neg AC$ and the forcing is required to not add new reals and be closed under a large family of automorphisms that change the obtained G while fixing the invariant A_G.

An analogous result is obtained for the embeddability relation $\sqsubseteq_{\mathsf{ArGp}}$ on countable Archimedean groups and its associated equivalence $\equiv_{\mathsf{ArGp}}$:

$$=^{\mathrm{cf}}_{\mathbb{R}} <_B \sqsubseteq_{\mathsf{ArGp}} <_B =^{2\text{-cf}}_{\mathbb{R}}$$

and

$$=^{+}_{\mathbb{R}} <_B \equiv_{\mathsf{ArGp}} <_B =^{2\text{-cf}}_{\mathbb{R}} \cap \left(=^{2\text{-cf}}_{\mathbb{R}}\right)^{-1}$$

Here, the asymmetric jump $(\cdot)^{\mathrm{cf}}$ and its iterates were defined by [Rosendal, C.; Cofinal families of Borel equivalence relations and quasiorders. J. Symbolic Logic **70** (2005), no. 4, 1325–1340. MR2194249]. As a consequence, countable sets of reals cannot be used to completely classify countable Archimedean groups up to bi-embeddability.

Archimedean circularly ordered groups (equivalently, "ordered" subgroups of S^1) are also considered, and the authors show that the corresponding isomorphism and bi-embeddability relations are both Borel bi-reducible with $=^+_{\mathbb{R}}$.

Finally, embeddability between colored linear orders is reduced to embeddability of countable ordered divisible Abelian groups and thus the latter relation is shown to be a complete analytic quasi-order. An application of a further reduction proves that (bi)-embeddability of real closed fields is also complete analytic.

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