Some lessons after the formalization of the ctm approach to forcing

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I would like to acknowledge

- Ken Kunen for his inspiring expositions. This formalization is a tiny homage to his memory.
- Larry Paulson for his work and encouragement.
- Mikhail Mandrykin and the rest of the Isabelle community for support.
- The organizers for their invitation.

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Tools

Agda, Coq, HOL Light, ACL2, Lean, Isabelle.
These support a variety of logics [Avigad, 2021].

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Notable formalization projects

- The Four Color Theorem [Gonthier, 2008], the odd order theorem [Gonthier et al., 2013].
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Ongoing formalization efforts

- Isabelle: HOL, AFP. See also Paulson [2018].
- Lean: `mathlib`-related projects promoted by Buzzard [Lean Community, 2021].

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Induction/recursion works at the object logic level: It is based on set-theoretical proofs of well-foundedness— of $\mathbb{N}$, of Ord, etc.

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- Quantifiers shift the indexing. The index indicates "how many quantifiers to jump."

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The Fundamental Theorems

Let $\langle \mathbb{P}, \preceq, \mathbb{1} \rangle \in M$ be a forcing notion. Given $G \subseteq \mathbb{P}$, we have
$M[G] := \{\mathrm{val}(P, G, \dot{a}) : \dot{a} \in M\}$
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Truth Lemma *for every* M*-generic* G,

$$M[G], [\mathrm{val}(P, G, \dot{a}_0), \ldots, \mathrm{val}(P, G, \dot{a}_n)] \models \varphi$$
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Density Lemma $p \Vdash \varphi\ [\dot{a}_0, \ldots, \dot{a}_n] \iff \{q \in \mathbb{P} : q \Vdash \varphi\ [\dot{a}_0, \ldots, \dot{a}_n]\}$ *is dense below* p.

The definition of forces

$$\texttt{forces\_eq}(p, t_1, t_2) := \forall s \in \mathrm{domain}(t_1) \cup \mathrm{domain}(t_2).\ \forall q \preceq p.$$
$$\texttt{forces\_mem}(q, s, t_1) \longleftrightarrow \texttt{forces\_mem}(q, s, t_2),$$
$$\texttt{forces\_mem}(p, t_1, t_2) := \forall v \preceq p.\ \exists q \preceq v.$$
$$\exists s.\ \exists r \in \mathbb{P}.\ \langle s, r \rangle \in t_2 \wedge q \preceq r \wedge \texttt{forces\_eq}(q, t_1, s)$$

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We codified this as a single recursion over the well-founded relation $frecR$.

$$\texttt{forces\_at}(\mathbb{P}, \preceq, \langle ftype, t_1, t_2, p\rangle) = 1 \iff \big(ftype = 0 \wedge (\cdots \text{forcing “}=\text{”} \cdots)\big) \vee \big(ftype = 1 \wedge (\cdots \text{forcing “}\in\text{”} \cdots)\big).$$

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lemma *forces_induction:*
assumes
"⋀τ ϑ. ⟦⋀σ. σ∈domain(ϑ) ⟹ Q(τ,σ)⟧ ⟹ R(τ,ϑ)"
"⋀τ ϑ. ⟦⋀σ. σ∈domain(τ) ∪ domain(ϑ) ⟹ R(σ,τ) ∧ R(σ,ϑ)⟧
⟹ Q(τ,ϑ)"
shows "Q(τ,ϑ) ∧ R(τ,ϑ)"

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- A model-theoretic rendition of forcing.
  - Proper extensions of ctms of ZF.
  - The construction of a ctm of $ZFC + \neg CH$ given one of ZFC.

```
theorem ctm_of_not_CH:
  assumes
    "M ≈ ω" "Transset(M)" "M ⊨ ZFC"
  shows
    "∃N.
      M ⊆ N ∧ N ≈ ω ∧ Transset(N) ∧ N ⊨ ZFC ∪ {·¬·CH··} ∧
      (∀α. Ord(α) ⟶ (α ∈ M ⟷ α ∈ N))"
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What can be deduced from the present formalization?

- Which instances of Separation and Replacement are needed to set the forcing machinery up.
- A verification that no choice is needed at the level of the object logic.

A new relativization discipline

relativize functional *"cardinal" "cardinal_rel"* **external**
relationalize *"cardinal_rel" "is_cardinal"*
synthesize *"is_cardinal"* **from_definition assuming** *"nonempty"*

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Recall: We can't perform induction on Isabelle terms! Done in ML.

39 replacement instances to rule them all

Combinatorics

- 7 instances for iteration of operations.
  - Fixpoints: lists (2), formulas (2).
  - nth element of a list (iteration of the tail operation).
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$$\psi(x, \alpha, y_1, \ldots, y_n) := \ulcorner \alpha = \min\{\beta \mid \exists \tau \in V_\beta .\ snd(x) \Vdash \varphi\ [fst(x), \tau, y_1, \ldots, y_n]\}\urcorner$$

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Forcing

- 2 instances for definitions by well-founded recursion (check names and forcing for atomic formulas).
- Replacement through $x \mapsto \check{x}$.
- Definition of $\dot{G}$.
- Name of choice functions in $M[G]$.
- 2 instances for the slalom needed for cardinal preservation by ccc forcing.

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Dependence of Separation on φ

The instances of Separation required for (1) grows recursively along the structure of φ.

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A **rule**:

$$[\![P(0)\ ;\ \bigwedge n.\ (P(n)\implies P(n+1))]\!]\implies \forall n\in \mathsf{nat}.\ P(n) \tag{2}$$

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After "applying" the rule (2) (`apply rule (2)`):

Proof state

1. $0^2 \geq 0.$
2. $\bigwedge n.\quad n^2 \geq 0 \implies (n+1)^2 \geq 0.$

Procedural proof of DC

```
theorem pointed_DC  : "(∀x∈A. ∃y∈A. ⟨x,y⟩∈ R) ⟹
                       ∀a∈A. (∃f ∈ nat→A. f`0 = a ∧ (∀n ∈ nat. ⟨f`n,f`succ(n)⟩∈R))"
  apply (rule)
  apply (insert AC_func_Pow)
  apply (drule allI)
  apply (drule_tac x="A" in spec)
  apply (drule_tac P="λf .∀x∈Pow(A) - {0}. f ` x ∈ x"
              and A="Pow(A)-{0}→ A"
              and Q=" ∃f∈nat→ A. f ` 0 = a ∧ (∀n∈nat. ⟨f ` n, f ` succ(n)⟩ ∈ R)" in bexE
  prefer 2 apply (assumption)
  apply (rename_tac s)
  apply (rule_tac x="λn∈nat. dc_witness(n,A,a,s,R)" in bexI)
   prefer 2  apply (blast intro:witness_funtype)
  apply (rule  conjI, simp)
  apply (rule ballI, rename_tac m)
  apply (subst beta, simp)+
  apply (rule witness_related, auto)
  done
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Not easy to read.

Goal

To provide a proof document, which can hopefully be read by mathematicians/set-theorists.

Isar: Forward proofs

lemma : $\forall x \in \{4, 5, 6\}.\ 0 < x$

Proof state

1 $\forall x \in \{4, 5, 6\}.\ 0 < x$

.

Isar: Forward proofs

lemma : $\forall x \in \{4, 5, 6\}.\ 0 < x$
proof

Proof state

1 $\bigwedge x.\ x \in \{4, 5, 6\} \implies 0 < x$

.

Isar: Forward proofs

lemma : $\forall x \in \{4,5,6\}.\ 0 < x$
proof
  fix x
  assume $x \in \{4,5,6\}$

Proof state

1 $\bigwedge x.\ x \in \{4,5,6\} \implies 0 < x$

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Isar: Forward proofs

lemma : $\forall x \in \{4, 5, 6\}.\ 0 < x$
proof
  fix x
  assume $x \in \{4, 5, 6\}$
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Proof state

1 $0 < x$

.

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lemma : $\forall x \in \{4, 5, 6\}.\ 0 < x$
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  fix x
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  then
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    by auto

Proof state

No subgoals!

.

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lemma : $\forall x \in \{4, 5, 6\}.\ 0 < x$
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Proof state

theorem $\forall x \in \{4, 5, 6\}.\ 0 < x$

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Proof state

theorem $\forall x \in \{4,5,6\}.\ 0 < x$

We now show some cherry-picked piece of the formalized math, and compare it with Kunen [2011].

Theorem IV.2.27 *Let M be a ctm for ZF, let $\mathbb{P} \in M$ be a forcing poset, and let G be $\mathbb{P}$-generic over M. Then $M[G] \models ZF$. Furthermore, $M[G] \models ZFC$ if $M \models ZFC$.*

For Power Set (similarly to Union above), it is sufficient to prove that whenever $a \in M[G]$, there is a $b \in M[G]$ such that $\mathcal{P}(a) \cap M[G] \subseteq b$. Fix $\tau \in M^{\mathbb{P}}$ such that $\tau_G = a$. Let $Q = (\mathcal{P}(\mathrm{dom}(\tau) \times \mathbb{P}))^M$. This is the set of all names $\vartheta \in M^{\mathcal{P}}$ such that $\mathrm{dom}(\vartheta) \subseteq \mathrm{dom}(\tau)$. Let $\pi = Q \times \{\mathbb{1}\}$ and let $b = \pi_G = \{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$.

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lemma Pow_inter_MG:
  assumes
    "a∈M[G]"
  shows
    "Pow(a) ∩ M[G] ∈ M[G]"
proof -
  from assms obtain τ where "τ ∈ M" "val(P,G, τ) = a"
    using GenExtD by auto
  let ?Q="Pow(domain(τ)×P) ∩ M"
  from ‹τ∈M›
  have "domain(τ)×P ∈ M" "domain(τ) ∈ M" [2 lines]
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$\{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$. Fix $\varkappa \in M^{\mathbb{P}}$ such that $\varkappa_G = c$, and let $\vartheta = \{\langle \sigma, p\rangle : \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \varkappa\}$; $\vartheta \in M$ by the Definability Lemma. Since $\vartheta \in Q$, we are done if we can show that $\vartheta_G = c$. $\vartheta_G \subseteq c$ holds because $\vartheta_G = \{\sigma_G : \exists p \in G\ p \Vdash \sigma \in \varkappa\}$ and all these σ_G lie in $\varkappa_G = c$ by the definition of $\Vdash$. To prove $c \subseteq \vartheta_G$: since $c \subseteq a = \tau_G$, every element of c is of the form σ_G for some $\sigma \in \mathrm{dom}(\tau)$. Since $\sigma_G \in c = \varkappa_G$, apply the Truth Lemma and fix $p \in G$ such that $p \Vdash \sigma \in \varkappa$; then $\langle \sigma, p\rangle \in \vartheta$, so $\sigma_G \in \vartheta_G$.

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    using GenExtI by simp
  have "Pow(a) ∩ M[G] ⊆ ?b"
  proof
    fix c
    assume "c ∈ Pow(a) ∩ M[G]"
    then obtain χ where "c∈M[G]" "χ ∈ M" "val(P,G,χ) = c"
      using GenExtD by auto
    let ?ϑ="{⟨σ,p⟩ ∈domain(τ)×P . p ⊩ ·0 ∈ 1· [σ,χ] }"
    have "arity(forces(Member(0,1))) = 6" [1 lines]
    with ‹domain(τ) ∈ M› ‹χ ∈ M›
    have "?ϑ ∈ M"
      using P_in_M one_in_M leq_in_M sats_fst_snd_in_M
      by simp
    then
    have "?ϑ ∈ ?Q" by auto
    then
    have "val(P,G,?ϑ) ∈ ?b"
      using one_in_G one_in_P generic val_of_elem [of ?ϑ one ?π G]
      by auto
    have "val(P,G,?ϑ) = c"
    proof(intro equalityI subsetI) [24 lines]
    next
      fix x
      assume "x ∈ c"
      with ‹c ∈ Pow(a) ∩ M[G]›
      have "x ∈ a" "c∈M[G]" "x∈M[G]"
        using transitivity_MG by auto
      with ‹val(P,G, τ) = a›
      obtain σ where "σ∈domain(τ)" "val(P,G,σ) = x"
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    using GenExtI by simp
  have "Pow(a) ∩ M[G] ⊆ ?b"
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      using one_in_G one_in_P generic val_of_elem [of ?ϑ one ?π G]
      by auto
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    proof(intro equalityI subsetI) [24 lines]
    next
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      assume "x ∈ c"
      with ‹c ∈ Pow(a) ∩ M[G]›
      have "x ∈ a" "c∈M[G]" "x∈M[G]"
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$\{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$. Fix $\varkappa \in M^{\mathbb{P}}$ such that $\varkappa_G = c$, and let $\vartheta = \{\langle \sigma, p\rangle : \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \varkappa\}$; $\vartheta \in M$ by the Definability Lemma. Since $\vartheta \in Q$, we are done if we can show that $\vartheta_G = c$. $\vartheta_G \subseteq c$ holds because $\vartheta_G = \{\sigma_G : \exists p \in G\ p \Vdash \sigma \in \varkappa\}$ and all these σ_G lie in $\varkappa_G = c$ by the definition of $\Vdash$. To prove $c \subseteq \vartheta_G$: since $c \subseteq a = \tau_G$, every element of c is of the form σ_G for some $\sigma \in \mathrm{dom}(\tau)$. Since $\sigma_G \in c = \varkappa_G$, apply the Truth Lemma and fix $p \in G$ such that $p \Vdash \sigma \in \varkappa$; then $\langle \sigma, p\rangle \in \vartheta$, so $\sigma_G \in \vartheta_G$.

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    using GenExtI by simp
  have "Pow(a) ∩ M[G] ⊆ ?b"
  proof
    fix c
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      using GenExtD by auto
    let ?ϑ="{⟨σ,p⟩ ∈domain(τ)×P . p ⊩ ·0 ∈ 1· [σ,χ] }"
    have "arity(forces(Member(0,1))) = 6" [1 lines]
    with ‹domain(τ) ∈ M› ‹χ ∈ M›
    have "?ϑ ∈ M"
      using P_in_M one_in_M leq_in_M sats_fst_snd_in_M
      by simp
    then
    have "?ϑ ∈ ?Q" by auto
    then
    have "val(P,G,?ϑ) ∈ ?b"
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Other approaches to set theory and forcing

- Lean: Full formalization of the Boolean-valued approach to forcing and the independence of CH [Han and van Doorn, 2020].

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A word on consistency strength

Isabelle/ZF	equiconsistent with ZF (?) [Paulson, 1989].
`ZFC_in_HOL`	approximately ZF + one inaccessible.
Lean (CiC)	ZF + ω inaccessibles [Carneiro, 2019].

Some mathporn (I)

$$\mathrm{forces}(0 \in 1) =$$

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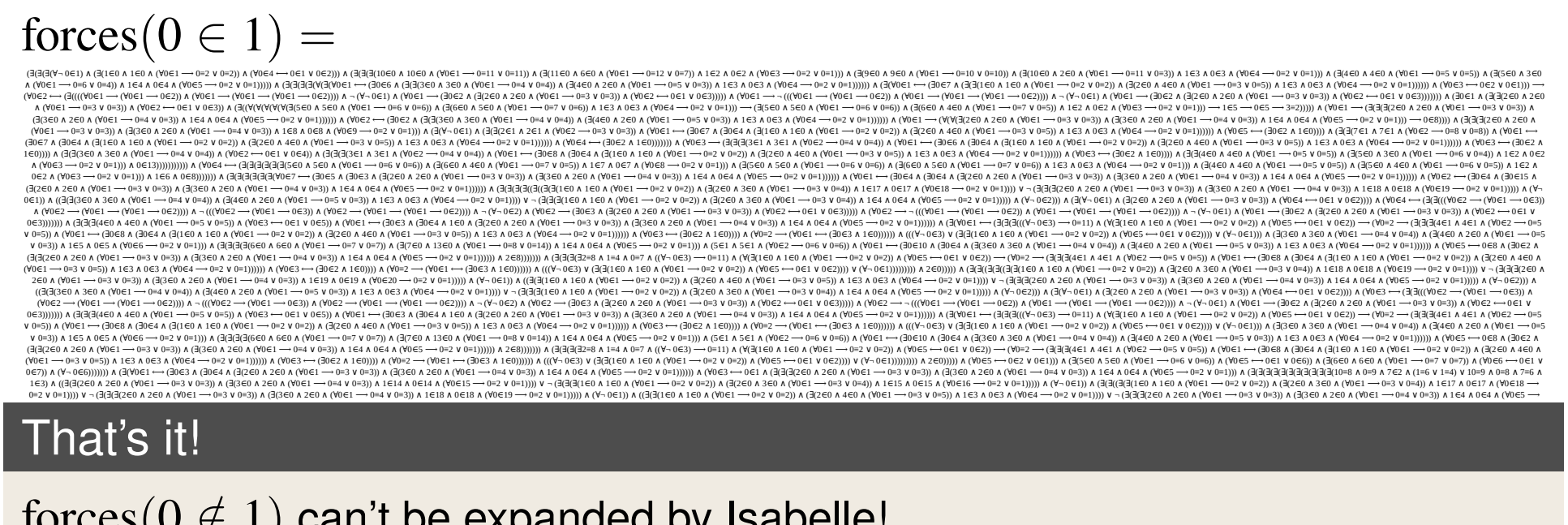
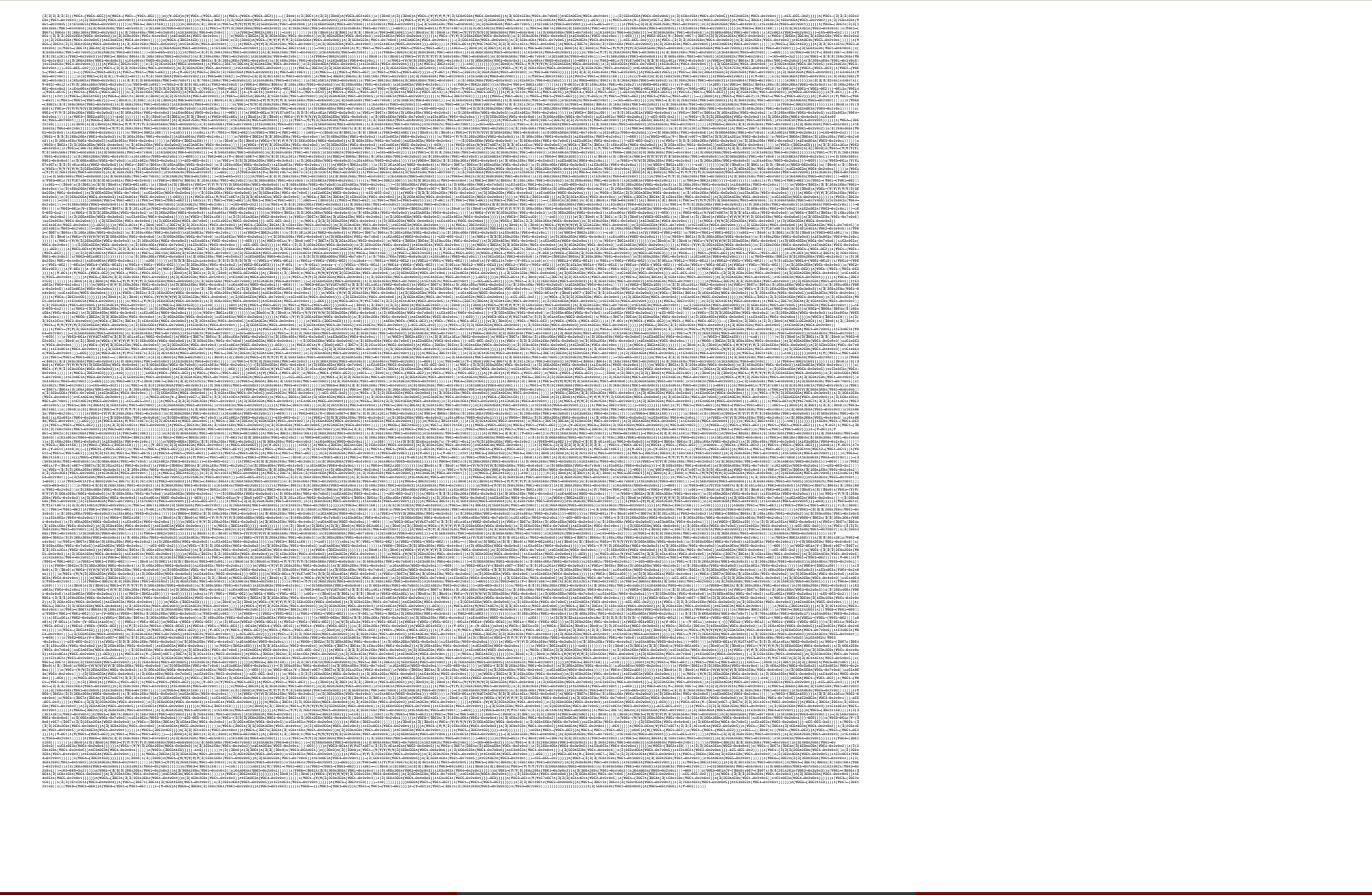


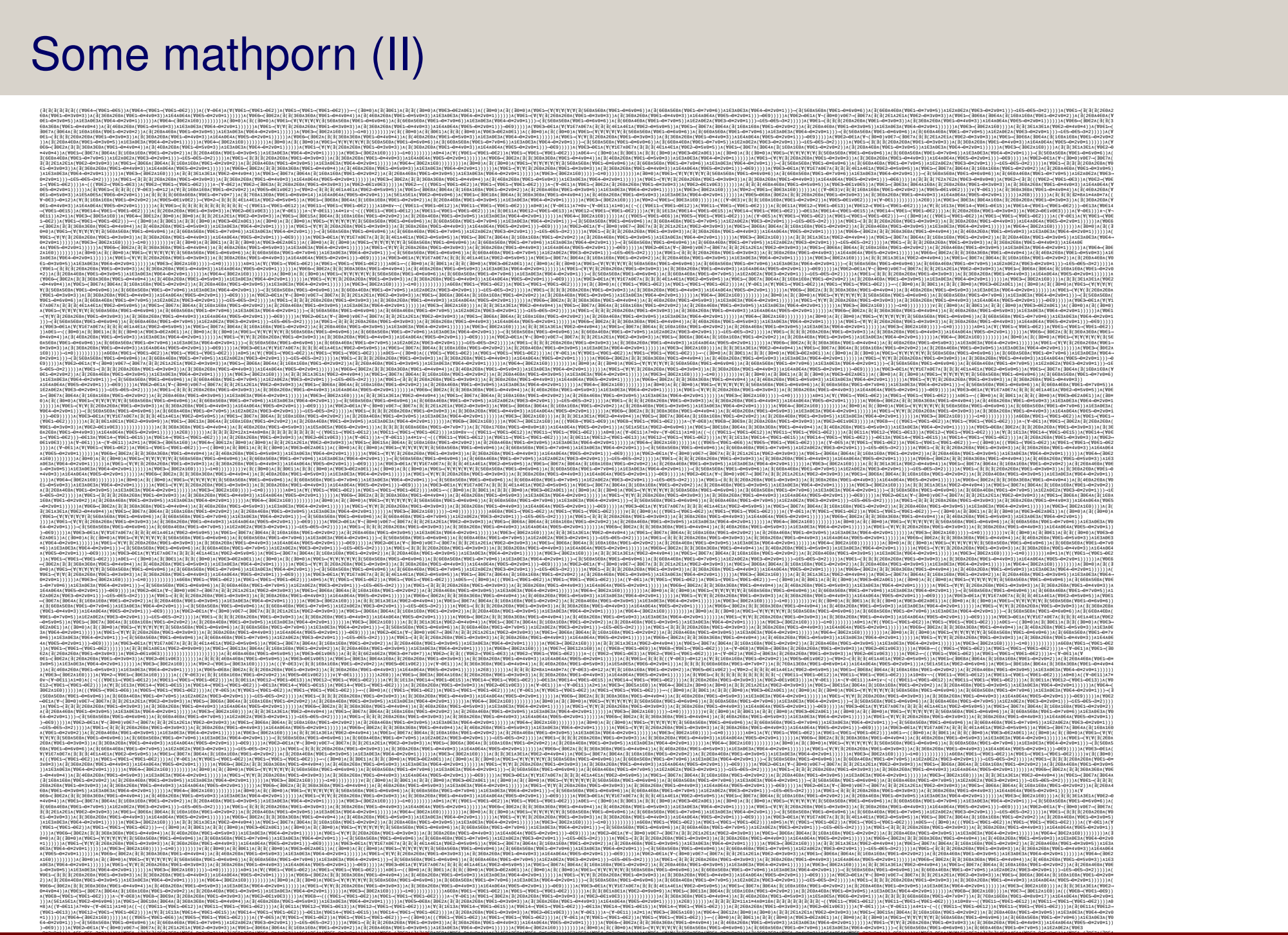
That's it!

$\mathrm{forces}(0 \notin 1)$ can't be expanded by Isabelle!

Some mathporn (II)



Some mathporn (II)

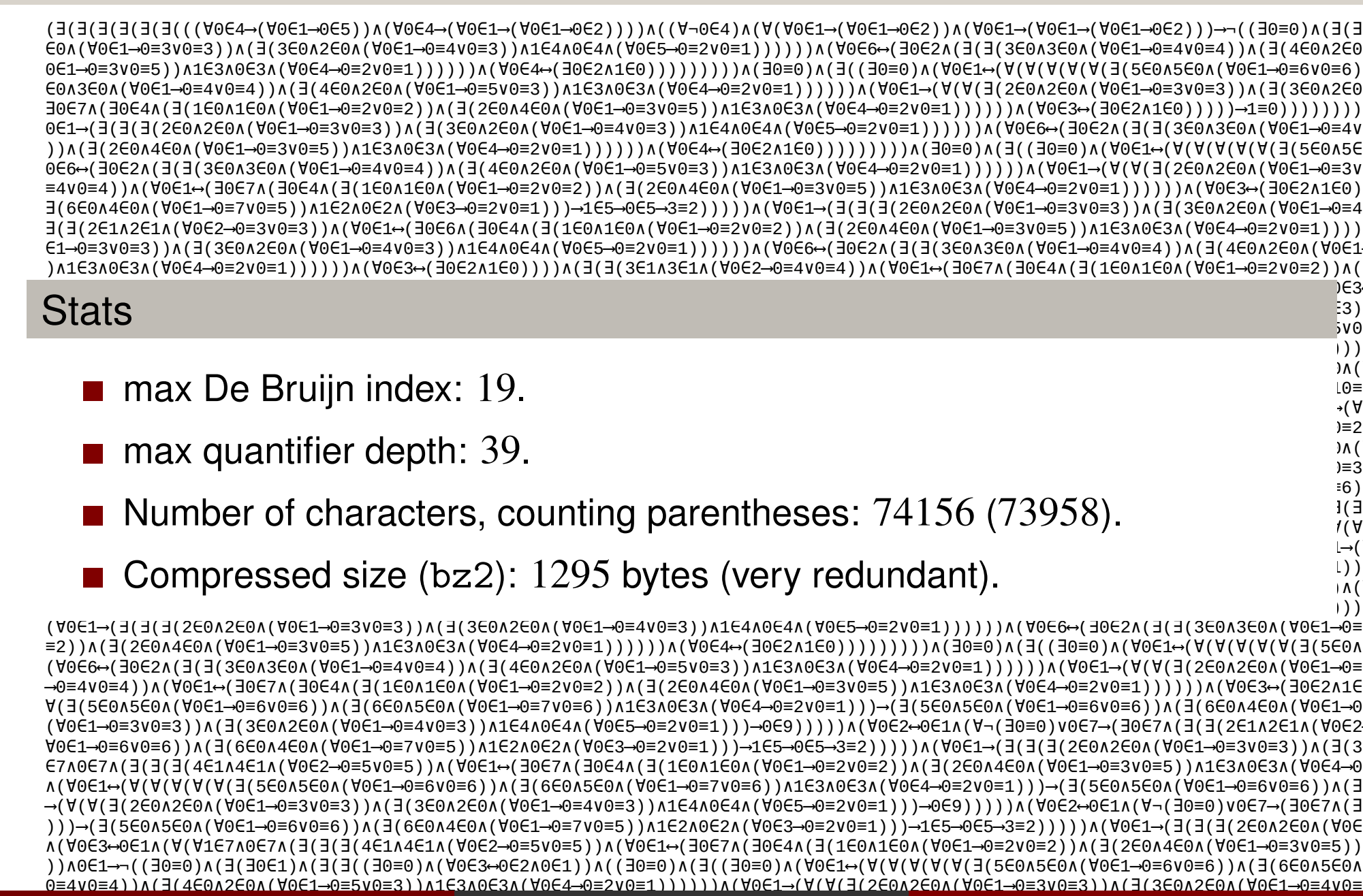


Some mathpori (II)

Some mathporn (II)

Some mathporn (II): CH

Some mathporn (II): CH



Stats

- max De Bruijn index: 19.
- max quantifier depth: 39.
- Number of characters, counting parentheses: 74156 (73958).
- Compressed size (bz2): 1295 bytes (very redundant).

How to read our work

Some links

- Site of the project: https://cs.famaf.unc.edu.ar/~pedro/forcing/
- Code repository: https://bitbucket.org/miguelpagano/forcing/src/master/src/

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Where to start

- Check out the outline which spares you from proofs. The last section, Main Definitions. . . , contains a summary.
- That section corresponds to the theory `Definitions_Main`, cross-linked to navigate the results.

Future work

On formalization

- Construction of ctms:
  - of ZFC from an inaccessible;
  - of the “forcing-abling” fragment, unconditionally.
- Collapse to obtain CH.
- Class forcing, using some definable $\mathbb{P} \subseteq M$ instead of $\mathbb{P} \in M$.
- Connect this development to Isabelle/HOL.

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More meta-theoretic mining

- Analyzing Separation instances in detail.
- Postpone uses of Powerset in ZF-Constructible.

```
— ‹Kunen IV.3.5›
lemma ccc_fun_approximation_lemma:
  notes le_trans[trans]
  assumes "cccᴹ(P,leq)" "A∈M" "B∈M" "f∈M[G]" "f : A → B"
  shows
    "∃F∈M. F : A → Pow(B) ∧ (∀a∈A. f`a ∈ F`a ∧ |F`a|ᴹ ≤ ω)"
proof -
  from ‹f∈M[G]›
  obtain f_dot where "f = val(P,G,f_dot)" "f_dot∈M" using GenE
  with assms
  obtain p where "p ⊩ ·0:1→2· [f_dot, Aᵛ, Bᵛ]" "p∈G" "p∈M"
```

Thanks! Obrigado! ¡Gracias!

```
lemma Aleph2_extension_le_conti
  includes G_generic_lemmas
  shows "ℵ₂^M[G] ≤ 2^(↑ℵ₀^M[G],M[G])"
proof -
  have "ℵ₂^M ∈ M[G]" "Ord(ℵ₂^M)"
  moreover from this
  have "ℵ₂^M ≲^M[G] ω →^M[G] 2" [4
  moreover from calculation
  have "ℵ₂^M ≲^M[G] |ω →^M[G] 2|^M[G]
  ultimately
  have "|ℵ₂^M|^M[G] ≤ 2^(↑ℵ₀^M[G],M[G])" [
  then
  show "ℵ₂^M[G] ≤ 2^(↑ℵ₀^M[G],M[G])" [4
qed
```

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