Bisimilarity is not Borel

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Labelled Transition Systems

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(Kripke frame, no propositional variables).

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Bisimulation & Bisimilarity

R is a *bisimulation* on LTS if it has the back-and-forth property: if $s\,R\,t$, then for all $a \in L$,

$$\forall s' : s \xrightarrow{a} s'.\exists t' (t \xrightarrow{a} t' \;\&\; s'\,R\,t') \text{ and the other way round}$$

s_1 is *bisimilar* to t_1 ($s_1 \sim t_1$) if there exists a bisimulation R such that $s_1\,R\,t_1$.

Logics for Bisimulation

Hennessy-Milner Logic (HML)

$$\varphi \equiv \top \mid \neg\varphi \mid \bigwedge_i \varphi_i \mid \langle a \rangle \psi$$

$$\mathbf{S}, s \models \langle a \rangle \psi \iff \exists s' : s \xrightarrow{a} s' \wedge \mathbf{S}, s' \models \psi$$

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Logical Characterization of Bisimulation

Two states in a LTS are bisimilar iff they satisfy the same HML formulas.

Labelled Markov Processes (LMP) and Non Determinism

LMP (Desharnais et al.), SKM (Doberkat)

$\langle S, \mathcal{S}, L, t\rangle$ such that $t_a(s) \in \mathbf{P}(\mathcal{S})$ for each $s \in S$ and $a \in L$, where

- $\langle S, \mathcal{S}\rangle$ is a measurable space;
- $\mathbf{P}(\mathcal{S})$ is the space of (sub)probability measures over $\langle S, \mathcal{S}\rangle$;
- $t_a : S \to \mathbf{P}(\mathcal{S})$ is measurable.

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NLMP (D'Argenio and Wolovick)

$\langle S, \mathcal{S}, L, T\rangle$ such that $T_a(s) \subseteq \mathbf{P}(S)$ para each $s \in S$ y $a \in L$, where:

- $\langle S, \mathcal{S}\rangle$, $\mathbf{P}(S)$ as before;
- For each s, $T_a(s)$ is a measurable set, i.e., $T_a : S \to \mathbf{P}(\mathcal{S})$.
- $T_a : S \to \mathbf{P}(\mathcal{S})$ is a measurable map.

A pinch of Descriptive Set Theory: Analytic Spaces

Definition

An *analytic* topological space is the continuous image of a Borel set (v.g., of reals).

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Examples

- The convex hull of a Borel set in $\mathbb{R}^n$;
- **The relation of isomorphism between countable structures.**

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Unique Structure Theorem

If a sub-σ-algebra $\mathcal{S} \subseteq \mathbf{B}(A)$ is countably generated and separates points, then it is $\mathbf{B}(A)$.

Logics for bisimulation on LMP

HML$_q$ (Larsen and Skou, Danos *et al.*)

$$\varphi \equiv \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle_q \varphi, \qquad q \in \mathbb{Q}$$

$$\mathbf{S}, s \models \langle a \rangle_q \psi \iff P(\{s' : s \xrightarrow{a} s' \;\&\; \mathbf{S}, s' \models \psi\}) > q$$

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Proof Strategy (D'Argenio, Celayes, PST)

This results holds for every process with an analytic state space and a logic $\mathscr{L}$ that satisfies: 1) $\mathscr{L}$ it contains $\top$ and $\wedge$; 2) for every $\varphi \in \mathscr{L}$, $[\![\varphi]\!]$ is measurable; 3) $\mathscr{L}$ is countable; and 4) $\mathscr{L}$ separates transitions "locally".

Completeness and some Counterexamples

Logical Characterization for image finite NLMP (D'Argenio *et. al*)

Two states in a image finite NLMP $\langle S, \mathcal{S}, L, t\rangle$ with $\langle S, \mathcal{S}\rangle$ analytic are bisimilar iff they satisfy the same $\mathscr{L}_f$ formulas:

$$\varphi \equiv \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a\rangle\{\varphi_i, p_i\}_{i=1}^{n}, \qquad p_i \in \mathbb{Q},\ n \in \mathbb{N}$$

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Analiticity is necessary

The category of LMP over arbitrary measurable spaces does not have the *right Ore property* (semipullbacks) and HML_q does not characterize bisimilarity (PST, *Inf& Comp.* **209** 2011)

At least image-countable is necessary

For NLMP over analytic spaces (D'Argenio, PST, Wolovick, *Math.Struct.Comp.Sci,* **22** 2009).

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One Desperate Question

Is there a countable logic for countable LTS?

An Elegant Answer (X. Caicedo)

There are at most $2^{\aleph_0} = \#\mathrm{Pow}(\mathbb{N})$ (bisimilarity classes) of countable LTS.
There is an injective f : bisimilarity classes $\to \mathrm{Pow}(\mathbb{N})$.
Choose arbitrary atomic "formulas" P_n ($n \in \mathbb{N}$) with the following semantics:

$$\mathbf{S}, s \models P_n \iff n \in f(\langle \mathbf{S}, s\rangle/\sim)$$

The logic $\mathscr{L}_X := \{P_n : n \in \mathbb{N}\}$ is sound and complete for bisimulation.

Proof Strategy

This results holds for every process with an analytic state space and a logic $\mathscr{L}$ that satisfies: 1) $\mathscr{L}$ it contains $\top$ and $\wedge$; 2) for every $\varphi \in \mathscr{L}$, $[\![\varphi]\!]$ is measurable; 3) $\mathscr{L}$ is countable; and 4) $\mathscr{L}$ separates transitions "locally".

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A notion of reduction

Definition

Let X_i be spaces and $A_i \subseteq X_i$ ($i = 1, 2$). A **reduction** of A_1 to A_2 is a map $f : X_1 \to X_2$ such that

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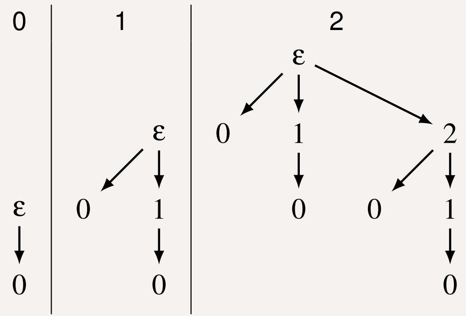
Example

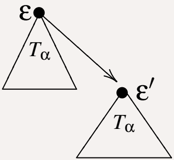
1. Poly-reductions of NP-problems.
2. Recursive reductions of undecidable problems.
3. Continuous reductions in topological spaces.

From WO to Trees. . .

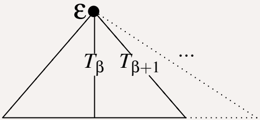


From *WO* to Trees. . .

- $T_0 \doteq \overset{\varepsilon}{\bullet}$;
- $T_{\alpha+1} \doteq$



- $T_\lambda \doteq$



$\beta < \lambda$.

WO^{α} "reduces" to $\sim$

Lemma

Let T_{α} be the tree associated to wellorders of type α. Then $T_{\alpha} \sim T_{\beta}$ iff $\alpha = \beta$.

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In general, for every linear order $\leq$ over $\mathbb{N}$ we can define its tree of finite decreasing chains.

Corollary

$WO^\alpha = (T_\cdot)^{-1}(T_\alpha/\sim)$.

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In general, for every linear order $\leq$ over $\mathbb{N}$ we can define its tree of finite decreasing chains.

Corollary

$WO^{\alpha} = (T_{\cdot})^{-1}(T_{\alpha}/\sim)$.

Theorem

The map $T_{\cdot}$ from is continuous, and hence $\sim$ is not Borel.

No Countable Borel Logic

Lemma

Assume $\mathscr{L}$ is a logic that characterizes bisimulation. Then

$$s/\!\sim\; = \bigcap\{[\![\varphi]\!] : S,s \models \varphi\} \cap \bigcap\{S \setminus [\![\varphi]\!] : S,s \not\models \varphi\}.$$

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Theorem

There is no countable Borel logic which is complete for $\sim$.

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Theorem

There is no countable Borel logic which is complete for $\sim$.

Proof. *If $\mathscr{L}$ is a countable logic that characterizes bisimulation and $[\![\mathscr{L}]\!] \subseteq \mathbf{B}(X)$, then $\langle \mathbf{S}, s\rangle / \sim$ is Borel. Moreover, the complexity of this Borel set is bounded. But the family WO^{α} has unbounded complexity (in the Borel hierarchy).*

¡Thank You!

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