Strict Refinement and Definability in Algebras

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Contents

The Refinement Property

$$A \cong B_1 \times B_2 \cong C_1 \times C_2$$

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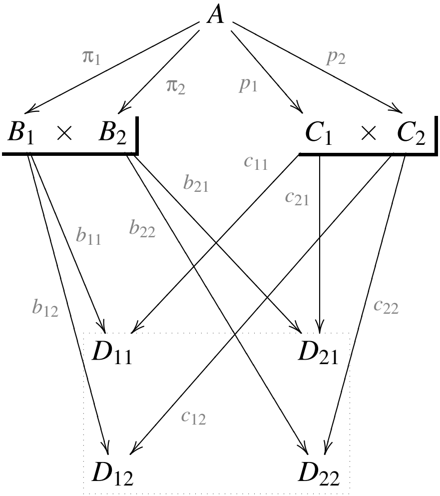
$\Rightarrow$ there exist D_{ij} such that

$$B_i \cong D_{i1} \times D_{i2} \text{ y } C_i \cong D_{1i} \times D_{2i}$$

with $i = 1, 2$

Strict Refinement (SRP)



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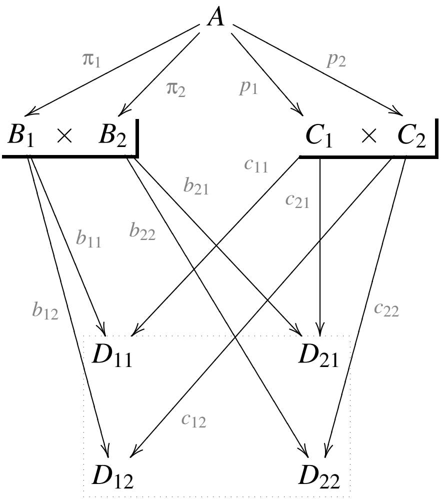

$$b_{11} \circ \pi_1 = c_{11} \circ p_1$$
$$b_{21} \circ \pi_2 = c_{21} \circ p_1$$
$$b_{12} \circ \pi_1 = c_{12} \circ p_1$$
$$\dots$$

Boolean Factor Congruences (BFC)

Given an algebra A, the set

$$FC(A) := \{\theta \in \mathrm{Con}\, A : \exists\theta^*.\theta \cap \theta^* = \Delta^A \text{ and } \theta \circ \theta^* = A \times A\}$$

is a distributive sublattice of $\mathrm{Con}\, A$.

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Theorem (Chang, Jónsson and Tarski)

SRP $\iff$ BFC $\iff$ factor congruences of direct products factorize (i.e., there are no skew factor congruences).

Central Elements and Factor Congruences

Variety with 0 & 1

There exist terms 0 and 1 such that

$$\mathcal{V} \models 0 = 1 \to x = y$$

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Example: Bounded lattices. Rings with identity.

$$\begin{array}{rcl} f : A & \longrightarrow & A_1 \times A_2 \\ x & \longmapsto & \langle x_1, x_2 \rangle \end{array}$$

$$\begin{array}{ccc} & A & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ A_1 & \times & A_2 \end{array}$$

- $(x, y) \in \ker \pi_1$

Central Elements and Factor Congruences

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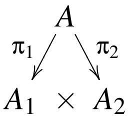

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- $(x, y) \in \ker \pi_1 \iff x_1 = y_1$;
- $x \cdot e = y \cdot e$, with $e = (1^B, 0^C) \rightsquigarrow$ λ-central elements (Vaggione)

Determining Property (DP)

Holds whenever central elements determine direct product descompositions:

$$\begin{aligned} Z &\longrightarrow FC(A)^2 \\ e &\longmapsto \langle \theta, \theta^* \rangle \end{aligned}$$

such that $e \; \theta \; 0$ and $e \; \theta^* \; 1$, is a bijection.

Definable Factor Congruences (DFC)

There exists a first order formula $\Phi(x,y,z)$ such that

$$B \times C \models \Phi\big(\langle a,b\rangle,\langle c,d\rangle,\langle 1,0\rangle\big) \text{ if and only if } a = c.$$

for all $B, C \in \mathcal{V}$, and $a, c \in B$, $b, d \in C$.

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Theorem (PST, Vaggione)

For a variety $\mathcal{V}$ with 0 & 1 TFAE:

1. *$\mathcal{V}$ has DFC;*
2. *$\mathcal{V}$ has BFC;*
3. *$\mathcal{V}$ has the DP.*

Hint for proof I

(DFC$\Rightarrow$DP) Immediate.

Hint for proof I

(DFC⇒DP) Immediate.

(BFC⇒DP) Two pairs of complementary factor congruences φ, φ^* and θ, θ^*.

$$0 \,\varphi\, e \,\varphi^*\, 1$$
$$0 \,\theta\, e \,\theta^*\, 1$$

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(DP⇒BFC) This one's really hard.

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To obtain a first order definition from DP we need terms. Let P be a property of varieties.

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Mal'cev Property

- $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n \subseteq \cdots \bigcup_n P_n = P$.
- P_n defined by a finite presentation in term-symbols and variables.

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- Congruence Permutability $\rightsquigarrow p(x,x,y) = p(y,x,x) = y$.

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Examples

- Congruence Permutability $\rightsquigarrow p(x,x,y) = p(y,x,x) = y$.
- Congruence Distributivity.
- BFC (Burris).

Devising factor congruences: The History

Problem

To work produce 'the most general situation' with two pairs of complementary factor congruences such that, e.g., $0 \; \theta \; z \; \theta^* \; 1$ y $0 \; \varphi \; z \; \varphi^* \; 1$.

Devising factor congruences: The History

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Idea (Vaggione)

Take $F := F_{\mathcal{V}}(X)$ with X very big:
$\forall p, q \in F\; \exists x_{pq} \in X : \;\; p\;\theta\;x_{pq}\;\theta^*\;q$.
Then take a good quotient so that $\theta \cap \theta^*$ and $\varphi \cap \varphi^*$ trivialize.

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1. (Willard) Recipe for a Mal’cev condition for BFC.

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Application

1. (Willard) Recipe for a Mal'cev condition for BFC.
2. (PST) Mal'cev condition for BFC and DFC.

Hint for proof III

BFC $\iff$ no skew factor congruences

I.e. each $\theta \in FC(A_1 \times A_2)$ is of the form $\theta_1 \times \theta_2$ with $\theta_i \in FC(A_i)$.

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Φ must be preserved by direct factors

$A_1 \times A_2 \models \Phi(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle, \langle e_1, e_2\rangle) \iff \langle x_1, x_2\rangle \theta \langle y_1, y_2\rangle \implies$
$x_1 \theta_1 y_1 \iff A_1 \models \Phi(x_1, y_1, e_1)$.

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$$\text{BFC} \xrightarrow{\text{Mal'cev}} \boxed{\text{Terms}} \longrightarrow \Phi(x, y, z)$$

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$$\text{BFC} \xrightarrow{\text{Mal'cev}} \boxed{\text{Terms}} \longrightarrow \Phi(x,y,z) \xrightarrow{\text{Pres }\mathbf{F}} \text{BFC}.$$

Same story with DFC!

Property (*)

Theorem (PST, Willard)

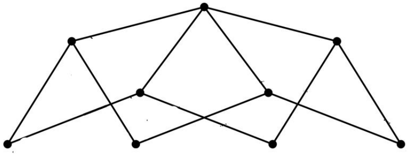
Let $\mathcal{V}$ be a variety. TFAE:

1. *There exists a first order formula $\pi(x,y,z,w)$ preserved by direct factors and products such that*
   1. $\mathcal{V} \models \pi(x,y,x,y)$
   2. $\mathcal{V} \models \pi(x,x,z,w)$
   3. $\mathcal{V} \models \pi(x,y,z,z) \to x = y$
2. *$\mathcal{V}$ has BFC.*

Problem

How complicated Φ might be?

Connected PO-groupoids



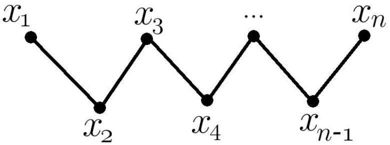
Definition

$\langle A, \wedge \rangle$ such that the relation

$$x \leq y \iff x \wedge y = x$$

is a partial order.

Connected PO-groupoids



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Connectivity

$\forall x, y \; \exists x_1, \ldots, x_n :$

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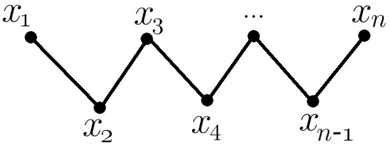

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$\forall x, y\ \exists x_1, \dots, x_n :$

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Connected posets have the SRP (CJT, McKenzie; cf. Hashimoto).

Connected PO-groupoids with 0 & 1

Theorem

Every semidegenerate variety $\mathcal{V}$ of connected PO-groupoids admits a Σ_4 formula Φ witnessing DFC.

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Proof: We may define a Π_2 formula $\psi(x,y,z)$ that satisfies

1. $\mathcal{V} \models \psi(x,y,x)$
2. $\mathcal{V} \models \psi(x,y,y)$
3. $\mathcal{V} \models \psi(x,x,z) \to x \leq z$

and using that one, π may be defined. With π and 0 & 1 we can define Φ.

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Corollary

If the language is finite, $\mathcal{V}_{DI}$ is Π_5.

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Thanks!

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$\mathcal{V}$ with factorable congruences and $\mathcal{V} = \Gamma^a(\mathcal{V}_{DI})$ imply $\mathcal{V}$ is a discriminator variety.
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Bonus Track: A Mal'cev Condition for BFC

$\boxed{|\alpha| = N}$

$$L_\alpha(\rho(\vec{X})) \approx R_\alpha(\rho(\vec{X}))$$
$$L_\alpha(\rho^*(\vec{X})) \approx R_\alpha(\rho^*(\vec{X}))$$

$$x \approx L_\varepsilon(\vec{X})$$
$$y \approx R_\varepsilon(\vec{X})$$
$$L_\varepsilon(\rho(\vec{X})) \approx L_1(\rho(\vec{X}))$$
$$R_j(\rho(\vec{X})) \approx L_{j+1}(\rho(\vec{X})) \qquad \text{if } 1 \le j \le N-1$$
$$R_N(\rho(\vec{X})) \approx R_\varepsilon(\rho(\vec{X}))$$

$\boxed{0 < |\alpha| < N}$

If $|\alpha|$ is even then

$$L_\alpha(\rho(\vec{X})) \approx L_{\alpha 1}(\rho(\vec{X}))$$
$$R_{\alpha j}(\rho(\vec{X})) \approx L_{\alpha(j+1)}(\rho(\vec{X})) \qquad \text{if } 1 \le j \le k-1$$
$$R_{\alpha k}(\rho(\vec{X})) \approx R_\alpha(\rho(\vec{X}))$$
$$L_\alpha(\rho^*(\vec{X})) \approx L_{\alpha(k+1)}(\rho^*(\vec{X}))$$
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$$R_{\alpha N}(\rho^*(\vec{X})) \approx R_\alpha(\rho^*(\vec{X}))$$

If $|\alpha|$ is odd then

$$L_\alpha(\sigma(\vec{X})) \approx L_{\alpha 1}(\sigma(\vec{X}))$$
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Bonus Track: Φ for PO-groupoids

$\psi(x,y,z) =$

$$\forall u_1,\dots,u_{2n-1}$$
$$\left(u_1\cdot x = u_1\cdot u_2 \wedge \left(\bigwedge_{i=2}^{n-1} u_{2i-1}\cdot u_{2i-2} = u_{2i-1}\cdot u_{2i}\right) \wedge\ u_{2n-1}\cdot u_{2n-2} = u_{2n-1}\cdot y\right) \rightarrow$$
$$\exists v_1,\dots,v_{n-1} : u_1\cdot x = u_1\cdot v_1 \wedge \left(\bigwedge_{i=2}^{n-1} u_{2i-1}\cdot v_{i-1} = u_{2i-1}\cdot v_i\right) \wedge\ u_{2n-1}\cdot v_{n-1} = u_{2n-1}\cdot z.$$

$$\Phi = \exists a_1,\dots,a_{n-1} : \pi(x,a_1,U_1(x,y,\vec{z}),U_1(x,y,\vec{w}))\wedge$$
$$\wedge \left(\bigwedge_{i\text{ odd}} \pi(a_i,a_{i+1},U_{i+1}(x,y,\vec{w}),U_{i+1}(x,y,\vec{z}))\right)\wedge$$
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$$\wedge\ \pi(a_{n-1},y,U_k(x,y,\vec{z}),U_k(x,y,\vec{w})).$$