Formalization of Forcing in Isabelle/ZF

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Summary

Why Isabelle/ZF?

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- Most advanced set theory formalized (around 2017).
- Structured proof language Isar [Wenzel, 1999].
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Cons

- A fraction of automation of Isabelle (`sledgehammer`, etc).
- “Untyped”, and too weak a metatheory.

Isabelle/ZF [Paulson and Grabczewski, 1996]

- An object logic of Isabelle axiomatized over the intuitionistic fragment *Pure* of higher order logic.
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- An object logic of Isabelle axiomatized over the intuitionistic fragment *Pure* of higher order logic.
- It postulates two types: `i` (sets) and `o` (booleans). Not inductively defined!
- The Replacement and Separation axiom schemes feature free high order variables.
- Induction/recursion is *internal* to the theory (it works as a layer on top of set-theoretical proofs of well-foundedness— of $\mathbb{N}$, of Ord, etc).

The ctm approach to forcing, 1/4

Countable transitive model (ctm) of ZF

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$\forall z.\ z \in M \longrightarrow (z \in x \longrightarrow z \in y)$, the **relativization** $\subseteq^M$ of $\subseteq$ to M.

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In this case, we know that $\subseteq$ is **absolute for transitive models**.

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Let $\langle \mathbb{P}, \preceq, \mathbb{1} \rangle \in M$ be a *forcing notion* (a preorder with top). Given an M*-generic* filter $G \subseteq \mathbb{P}$, we can adjoin it to M to form the **generic extension** $M[G]$.

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Every $a \in M[G]$ is coded by some $\dot{a} \in M$ through the function val:

$$M[G] := \{val(G, \dot{a}) : \dot{a} \in M\}$$

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Theorem ([Cohen, 1963])

There exists a formula-transformer $forces$ *such that for every* φ*,* M*-generic* G*, and* $\dot{a} \in M$*,*

$$M[G], [val(G, \dot{a})] \models \varphi \quad \Longleftrightarrow \quad \exists p \in G.\ M, [p, \preceq, \mathbb{P}, \dot{a}] \models forces(\varphi).$$

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$$\searrow \quad p \Vdash^{M}_{\mathbb{P},\preceq} \varphi(\dot{a}) \quad \swarrow$$

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3. Both M and $M[G]$ are standard (two-valued) models.
4. Ctms are used in an important fraction of the literature.

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Formalizing the independence of CH from the axioms of ZFC using ctms is one of the main goals of our project.

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A word on consistency strength

Isabelle/ZF + ctm	(far) less than ZF + one inaccessible.
`HOLZF`, `ZFC_in_HOL`	approximately ZF + one inaccessible.
Lean (CiC)	$ZF + \omega$ inaccessibles [Carneiro, 2019].

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3. We showed that generic extensions of ctms of ZF are also ctms of ZF (respectively, adding AC).
4. We provided the forcing notion that adds a Cohen real, therefore proving the existence of a nontrivial extension.

Some details

1. We adapted `ZF-Constructible` to obtain sharper absoluteness results.

This library originally contains a major development of relativization and absoluteness for *classes* $C :: \mathtt{i} \Rightarrow \mathtt{o}$, including the definition of the set `formula` :: `i` of internal formulas and model theoretic satisfaction.

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Around 40 absoluteness/closure lemmas now hold using weaker hypotheses on the class C (most of them, just that C is transitive and nonempty).

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We enhanced the recursion results of Isabelle/ZF as well as the relevant preservation results in ZF-Constructible, thus showing that forcing is absolute for atomic formulas.

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We may compare some of the code with the actual math [Kunen, 2011].

Theorem IV.2.27 *Let M be a ctm for ZF, let $\mathbb{P} \in M$ be a forcing poset, and let G be $\mathbb{P}$-generic over M. Then $M[G] \models ZF$. Furthermore, $M[G] \models ZFC$ if $M \models ZFC$.*

For Power Set (similarly to Union above), it is sufficient to prove that whenever $a \in M[G]$, there is a $b \in M[G]$ such that $\mathcal{P}(a) \cap M[G] \subseteq b$. Fix $\tau \in M^{\mathbb{P}}$ such that $\tau_G = a$. Let $Q = (\mathcal{P}(\text{dom}(\tau) \times \mathbb{P}))^M$. This is the set of all names $\vartheta \in M^{\mathcal{P}}$ such that $\text{dom}(\vartheta) \subseteq \text{dom}(\tau)$. Let $\pi = Q \times \{\mathbb{1}\}$ and let $b = \pi_G = \{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$.

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  assumes
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  shows
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proof -
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    assume "c ∈ Pow(a) ∩ M[G]"
    then obtain χ where "c∈M[G]" "χ ∈ M" "val(G,χ) = c"
      using GenExtD by auto
    let ?ϑ="{σp ∈domain(τ)×P . snd(σp) ⊩ (Member(0,1)) [fst(σp),χ] }"
    have "arity(forces(Member(0,1))) = 6" [1 lines]
```

Theorem IV.2.27 *Let M be a ctm for ZF, let $\mathbb{P} \in M$ be a forcing poset, and let G be $\mathbb{P}$-generic over M. Then $M[G] \models ZF$. Furthermore, $M[G] \models ZFC$ if $M \models ZFC$.*

For Power Set (similarly to Union above), it is sufficient to prove that whenever $a \in M[G]$, there is a $b \in M[G]$ such that $\mathcal{P}(a) \cap M[G] \subseteq b$. Fix $\tau \in M^{\mathbb{P}}$ such that $\tau_G = a$. Let $Q = (\mathcal{P}(\mathrm{dom}(\tau) \times \mathbb{P}))^M$. This is the set of all names $\vartheta \in M^{\mathcal{P}}$ such that $\mathrm{dom}(\vartheta) \subseteq \mathrm{dom}(\tau)$. Let $\pi = Q \times \{\mathbb{1}\}$ and let $b = \pi_G = \{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$.

```
lemma Pow_inter_MG:
  assumes
    "a∈M[G]"
  shows
    "Pow(a) ∩ M[G] ∈ M[G]"
proof -
  from assms obtain τ where "τ ∈ M" "val(G, τ) = a"
    using GenExtD by auto
  let ?Q="Pow(domain(τ)×P) ∩ M"
  from ‹τ∈M›
  have "domain(τ)×P ∈ M" "domain(τ) ∈ M" [2 lines]
  then
  have "?Q ∈ M" [17 lines]
  let ?π="?Q×{one}"
  let ?b="val(G,?π)"
  from ‹?Q∈M›
  have "?π∈M" [2 lines]
  then
  have "?b ∈ M[G]"
    using GenExtI by simp
  have "Pow(a) ∩ M[G] ⊆ ?b"
  proof
    fix c
    assume "c ∈ Pow(a) ∩ M[G]"
    then obtain χ where "c∈M[G]" "χ ∈ M" "val(G,χ) = c"
      using GenExtD by auto
    let ?ϑ="{σp ∈domain(τ)×P . snd(σp) ⊩ (Member(0,1)) [fst(σp),χ] }"
    have "arity(forces(Member(0,1))) = 6" [1 lines]
```

Theorem IV.2.27 *Let M be a ctm for ZF, let $\mathbb{P} \in M$ be a forcing poset, and let G be $\mathbb{P}$-generic over M. Then $M[G] \models ZF$. Furthermore, $M[G] \models ZFC$ if $M \models ZFC$.*

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```
lemma Pow_inter_MG:
  assumes
    "a∈M[G]"
  shows
    "Pow(a) ∩ M[G] ∈ M[G]"
proof -
  from assms obtain τ where "τ ∈ M" "val(G, τ) = a"
    using GenExtD by auto
  let ?Q="Pow(domain(τ)×P) ∩ M"
  from ‹τ∈M›
  have "domain(τ)×P ∈ M" "domain(τ) ∈ M" [2 lines]
  then
  have "?Q ∈ M" [17 lines]
  let ?π="?Q×{one}"
  let ?b="val(G,?π)"
  from ‹?Q∈M›
  have "?π∈M" [2 lines]
  then
  have "?b ∈ M[G]"
    using GenExtI by simp
  have "Pow(a) ∩ M[G] ⊆ ?b"
  proof
    fix c
    assume "c ∈ Pow(a) ∩ M[G]"
    then obtain χ where "c∈M[G]" "χ ∈ M" "val(G,χ) = c"
      using GenExtD by auto
    let ?ϑ="{σp ∈domain(τ)×P . snd(σp) ⊩ (Member(0,1)) [fst(σp),χ] }"
    have "arity(forces(Member(0,1))) = 6" [1 lines]
```

$\{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$. Fix $\varkappa \in M^{\mathbb{P}}$ such that $\varkappa_G = c$, and let $\vartheta = \{\langle\sigma, p\rangle : \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \varkappa\}$; $\vartheta \in M$ by the Definability Lemma. Since $\vartheta \in Q$, we are done if we can show that $\vartheta_G = c$. $\vartheta_G \subseteq c$ holds because $\vartheta_G = \{\sigma_G : \exists p \in G\ p \Vdash \sigma \in \varkappa\}$ and all these σ_G lie in $\varkappa_G = c$ by the definition of $\Vdash$. To prove $c \subseteq \vartheta_G$: since $c \subseteq a = \tau_G$, every element of c is of the form σ_G for some $\sigma \in \mathrm{dom}(\tau)$. Since $\sigma_G \in c = \varkappa_G$, apply the Truth Lemma and fix $p \in G$ such that $p \Vdash \sigma \in \varkappa$; then $\langle\sigma, p\rangle \in \vartheta$, so $\sigma_G \in \vartheta_G$.

```
    using GenExtI by simp
  have "Pow(a) ∩ M[G] ⊆ ?b"
  proof
    fix c
    assume "c ∈ Pow(a) ∩ M[G]"
    then obtain χ where "c∈M[G]" "χ ∈ M" "val(G,χ) = c"
      using GenExtD by auto
    let ?ϑ="{σp ∈domain(τ)×P . snd(σp) ⊩ (Member(0,1)) [fst(σp),χ] }"
    have "arity(forces(Member(0,1))) = 6" [1 lines]
    with ‹domain(τ) ∈ M› ‹χ ∈ M›
    have "?ϑ ∈ M"
      using P_in_M one_in_M leq_in_M sats_fst_snd_in_M
      by simp
    then
    have "?ϑ ∈ ?Q" by auto
    then
    have "val(G,?ϑ) ∈ ?b"
      using one_in_G one_in_P generic val_of_elem [of ?ϑ one ?π G]
      by auto
    have "val(G,?ϑ) = c"
    proof(intro equalityI subsetI) [24 lines]
    next
      fix x
      assume "x ∈ c"
      with ‹c ∈ Pow(a) ∩ M[G]›
      have "x ∈ a" "c∈M[G]" "x∈M[G]"
        using transitivity_MG by auto
```

$\{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$. Fix $\varkappa \in M^{\mathbb{P}}$ such that $\varkappa_G = c$, and let $\vartheta = \{\langle \sigma, p\rangle : \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \varkappa\}$; $\vartheta \in M$ by the Definability Lemma. Since $\vartheta \in Q$, we are done if we can show that $\vartheta_G = c$. $\vartheta_G \subseteq c$ holds because $\vartheta_G = \{\sigma_G : \exists p \in G\ p \Vdash \sigma \in \varkappa\}$ and all these σ_G lie in $\varkappa_G = c$ by the definition of $\Vdash$. To prove $c \subseteq \vartheta_G$: since $c \subseteq a = \tau_G$, every element of c is of the form σ_G for some $\sigma \in \mathrm{dom}(\tau)$. Since $\sigma_G \in c = \varkappa_G$, apply the Truth Lemma and fix $p \in G$ such that $p \Vdash \sigma \in \varkappa$; then $\langle \sigma, p\rangle \in \vartheta$, so $\sigma_G \in \vartheta_G$.

```
    using GenExtI by simp
  have "Pow(a) ∩ M[G] ⊆ ?b"
  proof
    fix c
    assume "c ∈ Pow(a) ∩ M[G]"
    then obtain χ where "c∈M[G]" "χ ∈ M" "val(G,χ) = c"
      using GenExtD by auto
    let ?ϑ="{σp ∈domain(τ)×P . snd(σp) ⊩ (Member(0,1)) [fst(σp),χ] }"
    have "arity(forces(Member(0,1))) = 6" [1 lines]
    with ‹domain(τ) ∈ M› ‹χ ∈ M›
    have "?ϑ ∈ M"
      using P_in_M one_in_M leq_in_M sats_fst_snd_in_M
      by simp
    then
    have "?ϑ ∈ ?Q" by auto
    then
    have "val(G,?ϑ) ∈ ?b"
      using one_in_G one_in_P generic val_of_elem [of ?ϑ one ?π G]
      by auto
    have "val(G,?ϑ) = c"
    proof(intro equalityI subsetI) [24 lines]
    next
      fix x
      assume "x ∈ c"
      with ‹c ∈ Pow(a) ∩ M[G]›
      have "x ∈ a" "c∈M[G]" "x∈M[G]"
        using transitivity_MG by auto
```

$\{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$. Fix $\varkappa \in M^{\mathbb{P}}$ such that $\varkappa_G = c$, and let $\vartheta = \{\langle \sigma, p\rangle : \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \varkappa\}$; $\vartheta \in M$ by the Definability Lemma. Since $\vartheta \in Q$, we are done if we can show that $\vartheta_G = c$. $\vartheta_G \subseteq c$ holds because $\vartheta_G = \{\sigma_G : \exists p \in G\ p \Vdash \sigma \in \varkappa\}$ and all these σ_G lie in $\varkappa_G = c$ by the definition of $\Vdash$. To prove $c \subseteq \vartheta_G$: since $c \subseteq a = \tau_G$, every element of c is of the form σ_G for some $\sigma \in \mathrm{dom}(\tau)$. Since $\sigma_G \in c = \varkappa_G$, apply the Truth Lemma and fix $p \in G$ such that $p \Vdash \sigma \in \varkappa$; then $\langle \sigma, p\rangle \in \vartheta$, so $\sigma_G \in \vartheta_G$.

```
    using GenExtI by simp
  have "Pow(a) ∩ M[G] ⊆ ?b"
  proof
    fix c
    assume "c ∈ Pow(a) ∩ M[G]"
    then obtain χ where "c∈M[G]" "χ ∈ M" "val(G,χ) = c"
      using GenExtD by auto
    let ?ϑ="{σp ∈domain(τ)×P . snd(σp) ⊩ (Member(0,1)) [fst(σp),χ] }"
    have "arity(forces(Member(0,1))) = 6" [1 lines]
    with ‹domain(τ) ∈ M› ‹χ ∈ M›
    have "?ϑ ∈ M"
      using P_in_M one_in_M leq_in_M sats_fst_snd_in_M
      by simp
    then
    have "?ϑ ∈ ?Q" by auto
    then
    have "val(G,?ϑ) ∈ ?b"
      using one_in_G one_in_P generic val_of_elem [of ?ϑ one ?π G]
      by auto
    have "val(G,?ϑ) = c"
    proof(intro equalityI subsetI) [24 lines]
    next
      fix x
      assume "x ∈ c"
      with ‹c ∈ Pow(a) ∩ M[G]›
      have "x ∈ a" "c∈M[G]" "x∈M[G]"
        using transitivity_MG by auto
```

$\{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$. Fix $\varkappa \in M^{\mathbb{P}}$ such that $\varkappa_G = c$, and let $\vartheta = \{\langle\sigma, p\rangle : \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \varkappa\}$; $\vartheta \in M$ by the Definability Lemma. Since $\vartheta \in Q$, we are done if we can show that $\vartheta_G = c$. $\vartheta_G \subseteq c$ holds because $\vartheta_G = \{\sigma_G : \exists p \in G\ p \Vdash \sigma \in \varkappa\}$ and all these σ_G lie in $\varkappa_G = c$ by the definition of $\Vdash$. To prove $c \subseteq \vartheta_G$: since $c \subseteq a = \tau_G$, every element of c is of the form σ_G for some $\sigma \in \mathrm{dom}(\tau)$. Since $\sigma_G \in c = \varkappa_G$, apply the Truth Lemma and fix $p \in G$ such that $p \Vdash \sigma \in \varkappa$; then $\langle\sigma, p\rangle \in \vartheta$, so $\sigma_G \in \vartheta_G$.

```
    using GenExtI by simp
  have "Pow(a) ∩ M[G] ⊆ ?b"
  proof
    fix …
    a… ∈ Pow(a) ∩ M[G]"
    …btain χ where "c∈M[G]" "χ ∈ M" "val(G,χ) = c"
      then obtain χ where "c∈M[G]" "χ ∈ M" "val(G,χ) = c"
        using GenExtD by auto
      let ?ϑ="{<σ,p> ∈domain(τ)×P . p ⊩ (Member(0,1)) [σ,χ] }"
      have "arity(forces(Member(0,1))) = 6"
        using arity_forces_at by auto
      using … one_in_M … sats_fst_snd_in_M
      by simp
    then
    have "?ϑ ∈ ?Q" by auto
    then
    have "val(G,?ϑ) ∈ ?b"
      using one_in_G one_in_P generic val_of_elem [of ?ϑ one ?π G]
      by auto
    have "val(G,?ϑ) = c"
    proof(intro equalityI subsetI) [24 lines]
    next
      fix x
      assume "x ∈ c"
      with ‹c ∈ Pow(a) ∩ M[G]›
      have "x ∈ a" "c∈M[G]" "x∈M[G]"
        using transitivity_MG by auto
```

$\{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$. Fix $\varkappa \in M^{\mathbb{P}}$ such that $\varkappa_G = c$, and let $\vartheta = \{\langle\sigma, p\rangle : \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \varkappa\}$; $\vartheta \in M$ by the Definability Lemma. Since $\vartheta \in Q$, we are done if we can show that $\vartheta_G = c$. $\vartheta_G \subseteq c$ holds because $\vartheta_G = \{\sigma_G : \exists p \in G\ p \Vdash \sigma \in \varkappa\}$ and all these σ_G lie in $\varkappa_G = c$ by the definition of $\Vdash$. To prove $c \subseteq \vartheta_G$: since $c \subseteq a = \tau_G$, every element of c is of the form σ_G for some $\sigma \in \mathrm{dom}(\tau)$. Since $\sigma_G \in c = \varkappa_G$, apply the Truth Lemma and fix $p \in G$ such that $p \Vdash \sigma \in \varkappa$; then $\langle\sigma, p\rangle \in \vartheta$, so $\sigma_G \in \vartheta_G$.

```
   using GenExtI by simp
 have "Pow(a) ∩ M[G] ⊆ ?b"
 proof
   fix c
   assume "c ∈ Pow(a) ∩ M[G]"
   then obtain χ where "c∈M[G]" "χ ∈ M" "val(G,χ) = c"
     using GenExtD by auto
   let ?ϑ="{σp ∈domain(τ)×P . snd(σp) ⊩ (Member(0,1)) [fst(σp),χ] }"
   have "arity(forces(Member(0,1))) = 6" [1 lines]
   with ‹domain(τ) ∈ M› ‹χ ∈ M›
   have "?ϑ ∈ M"
     using P_in_M one_in_M leq_in_M sats_fst_snd_in_M
     by simp
   then
   have "?ϑ ∈ ?Q" by auto
   then
   have "val(G,?ϑ) ∈ ?b"
     using one_in_G one_in_P generic val_of_elem [of ?ϑ one ?π G]
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   have "val(G,?ϑ) = c"
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   next
     fix x
     assume "x ∈ c"
     with ‹c ∈ Pow(a) ∩ M[G]›
     have "x ∈ a" "c∈M[G]" "x∈M[G]"
       using transitivity_MG by auto
```

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```
    using GenExtI by simp
  have "Pow(a) ∩ M[G] ⊆ ?b"
  proof
    fix c
    assume "c ∈ Pow(a) ∩ M[G]"
    then obtain χ where "c∈M[G]" "χ ∈ M" "val(G,χ) = c"
      using GenExtD by auto
    let ?ϑ="{σp ∈domain(τ)×P . snd(σp) ⊩ (Member(0,1)) [fst(σp),χ] }"
    have "arity(forces(Member(0,1))) = 6" [1 lines]
    with ‹domain(τ) ∈ M› ‹χ ∈ M›
    have "?ϑ ∈ M"
      using P_in_M one_in_M leq_in_M sats_fst_snd_in_M
      by simp
    then
    have "?ϑ ∈ ?Q" by auto
    then
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      using one_in_G one_in_P generic val_of_elem [of ?ϑ one ?π G]
      by auto
    have "val(G,?ϑ) = c"
    proof(intro equalityI subsetI) [24 lines]
    next
      fix x
      assume "x ∈ c"
      with ‹c ∈ Pow(a) ∩ M[G]›
      have "x ∈ a" "c∈M[G]" "x∈M[G]"
        using transitivity_MG by auto
      with ‹val(G, χ) = a›
```

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Looking forward

Formalizing math

- Cofinality, Kőnig's Theorem, Shanin's Δ-system Lemma.
- Forcing notion for adding κ Cohen reals.
- Theorems on preservation of cardinals.

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- Theorems on preservation of cardinals.

Technical aids

- Automatic relativization and proof of absoluteness of concepts.
- "Relative functions" (e.g., $\mathscr{P}^M$, $|\cdot|^M$, cf^M).

Thank you!

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Extra: Locale structure involving set models

`forcing_notion`	=	preorder $\mathbb{P}$ with top.
`M_ZF_trans`	=	set model M of the ZF axioms + M transitive
`M_ctm`	=	`M_ZF_trans` + M countable
`forcing_data`	=	`M_ctm` + `forcing_notion` $\mathbb{P} \in M$
`separative_notion`	=	`forcing_notion` + $\mathbb{P}$ separative
`M_ctm_separative`	=	`forcing_data` + `separative_notion`
`G_generic`	=	`forcing_data` + G is M-generic

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Extra: More code review

We only show the second inclusion $c \subseteq \vartheta_G = val(G, \vartheta)$ (the first one is proved in the course of the 24 folded lines).

$\{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$. Fix $\varkappa \in M^{\mathbb{P}}$ such that $\varkappa_G = c$, and let $\vartheta = \{\langle\sigma, p\rangle : \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \varkappa\}$; $\vartheta \in M$ by the Definability Lemma. Since $\vartheta \in Q$, we are done if we can show that $\vartheta_G = c$. $\vartheta_G \subseteq c$ holds because $\vartheta_G = \{\sigma_G : \exists p \in G\ p \Vdash \sigma \in \varkappa\}$ and all these σ_G lie in $\varkappa_G = c$ by the definition of $\Vdash$. To prove $c \subseteq \vartheta_G$: since $c \subseteq a = \tau_G$, every element of c is of the form σ_G for some $\sigma \in \mathrm{dom}(\tau)$. Since $\sigma_G \in c = \varkappa_G$, apply the Truth Lemma and fix $p \in G$ such that $p \Vdash \sigma \in \varkappa$; then $\langle\sigma, p\rangle \in \vartheta$, so $\sigma_G \in \vartheta_G$.

```
    have "val(G,?ϑ) = c"
    proof(intro equalityI subsetI) [24 lines]
    next
      fix x
      assume "x ∈ c"
      with ‹c ∈ Pow(a) ∩ M[G]›
      have "x ∈ a" "c∈M[G]" "x∈M[G]"
        using transitivity_MG by auto
      with ‹val(G, τ) = a›
      obtain σ where "σ∈domain(τ)" "val(G,σ) = x"
        using elem_of_val by blast
      moreover note ‹x∈c› ‹val(G,χ) = c›
      moreover from calculation
      have "val(G,σ) ∈ val(G,χ)"
        by simp
      moreover note ‹c∈M[G]› ‹x∈M[G]›
      moreover from calculation
      have "sats(M[G],Member(0,1),[x,c])"
        by simp
      moreover
      have "σ∈M" [8 lines]
      moreover
      note ‹χ ∈ M›
      ultimately
      obtain p where "p∈G" "(p ⊩ Member(0,1) [σ,χ])"
        using generic truth_lemma[of "Member(0,1)" "G" "[σ,χ]" ] nat_simp_union
        by auto
```

$\{\vartheta_G : \vartheta \in Q\}$. Now, consider any $c \in \mathcal{P}(a) \cap M[G]$; we need to show that $c \in b$. Fix $\varkappa \in M^{\mathbb{P}}$ such that $\varkappa_G = c$, and let $\vartheta = \{\langle\sigma, p\rangle : \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \varkappa\}$; $\vartheta \in M$ by the Definability Lemma. Since $\vartheta \in Q$, we are done if we can show that $\vartheta_G = c$. $\vartheta_G \subseteq c$ holds because $\vartheta_G = \{\sigma_G : \exists p \in G\ p \Vdash \sigma \in \varkappa\}$ and all these σ_G lie in $\varkappa_G = c$ by the definition of $\Vdash$. To prove $c \subseteq \vartheta_G$: since $c \subseteq a = \tau_G$, every element of c is of the form σ_G for some $\sigma \in \mathrm{dom}(\tau)$. Since $\sigma_G \in c = \varkappa_G$, apply the Truth Lemma and fix $p \in G$ such that $p \Vdash \sigma \in \varkappa$; then $\langle\sigma, p\rangle \in \vartheta$, so $\sigma_G \in \vartheta_G$.

```
  have "val(G,?ϑ) = c"
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  next
    fix x
    assume "x ∈ c"
    with ‹c ∈ Pow(a) ∩ M[G]›
    have "x ∈ a" "c∈M[G]" "x∈M[G]"
      using transitivity_MG by auto
    with ‹val(G, τ) = a›
    obtain σ where "σ∈domain(τ)" "val(G,σ) = x"
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    moreover note ‹x∈c› ‹val(G,χ) = c›
    moreover from calculation
    have "val(G,σ) ∈ val(G,χ)"
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    note ‹χ ∈ M›
    ultimately
    obtain p where "p∈G" "(p ⊩ Member(0,1) [σ,χ])"
      using generic truth_lemma[of "Member(0,1)" "G" "[σ,χ]" ] nat_simp_union
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  have "val(G,?ϑ) = c"
  proof(intro equalityI subsetI) [24 lines]
  next
    fix x
    assume "x ∈ c"
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    with ‹val(G,σ) =  x› ‹p∈G›
    show "x∈val(G,?ϑ)"
      using val_of_elem [of _ _ "?ϑ"] by auto
  qed
  with ‹val(G,?ϑ) ∈ ?b›
  show "c∈?b" by simp
qed
then
have "Pow(a) ∩ M[G] = {x∈?b . x⊆a ∧ x∈M[G]}"
  by auto
also from ‹a∈M[G]›
have " ... = {x∈?b . (M[G], [x,a] ⊨ subset_fm(0,1)) ∧ x∈M[G]}"
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