Regressive order on subsets of regular cardinals

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Summary

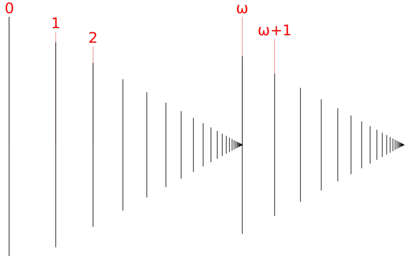
1. Intro: The club filter on ω_1
   - Club sets
   - Stationary sets
   - Pressing-down lemma
2. The regressive order on $[\omega_1]^{\aleph_1}$
   - Problems and examples
   - Lower bounds
   - Characterization of $<_R$ for ω_1
3. Generalizations
   - $<\beta$-to-one regressive maps
   - Many maximal elements

The club filter on ω_1

Let ω_1 be the first uncountable ordinal.

$C \subseteq \omega_1$ is club (*closed unbounded*) if it is unbounded in ω_1 and it contains all of its limit points.

Analogy: Borel sets of measure 1 in $[0,1]$.



Example

- The set Lim of limit ordinals in ω_1: $\{\omega, \omega \cdot 2, \omega \cdot 3, \ldots, \omega^2, \omega^2 + \omega, \ldots\}$
- Given $g : \omega_1 \to \omega_1$, $C_g := \{\beta \in \omega_1 : \forall \alpha < \beta (g(\alpha) < \beta)\}$

Lemma

Clubs are closed under countable intersections.

Hence subsets containing a club form a filter, the club filter.

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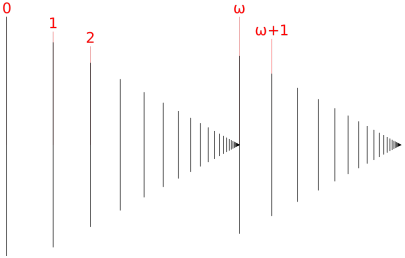

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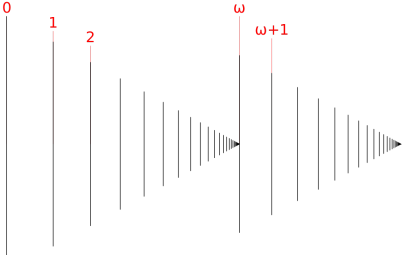

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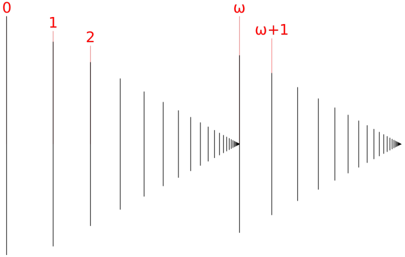

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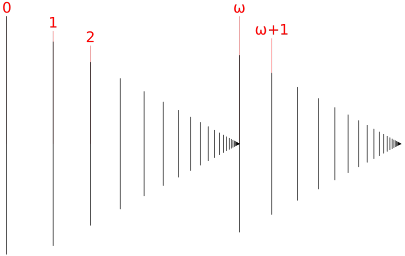

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Stationary sets

$N \subseteq \omega_1$ is nonstationary if its complement contains a club. They form an ideal, dual to the club filter.

Analogy: sets of outer measure 0 in $[0,1]$.

Example

$N_0 := \bigcup\{(\delta, \delta+\omega] : \delta \in \omega_1 \text{ limit}\} \subseteq \omega_1 \setminus \{\omega^\alpha : 2 \leq \alpha \in \omega_1\}$.

$S \subseteq \omega_1$ is stationary if it is not nonstationary. Equivalently, S intersects every club.

Analogy: sets of positive outer measure in $[0,1]$.

Some properties

- Every stationary set is unbounded in ω_1 (intersects every $[\alpha, \omega_1)$).
- Every stationary set contains (many) limit ordinals.

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The pressing-down lemma

Intuition

- Sets in the club filter have “density 1 at infinity.”
- Stationary sets have “positive density at infinity”

We can't bring a stationary set from infinity in a 1-1 fashion.

Fodor's Lemma

Let $S \subseteq \omega_1$ be stationary and $f : S \to \omega_1$ such that $f(\alpha) < \alpha$ for all $\alpha \in S$. Then there exists $\beta \in \omega_1$ such that $f^{-1}(\beta)$ is stationary (viz., uncountable).

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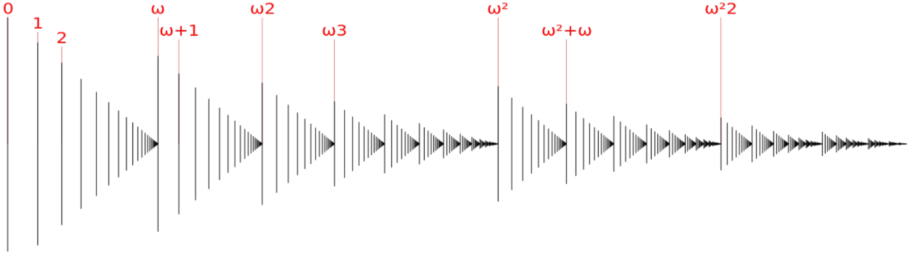

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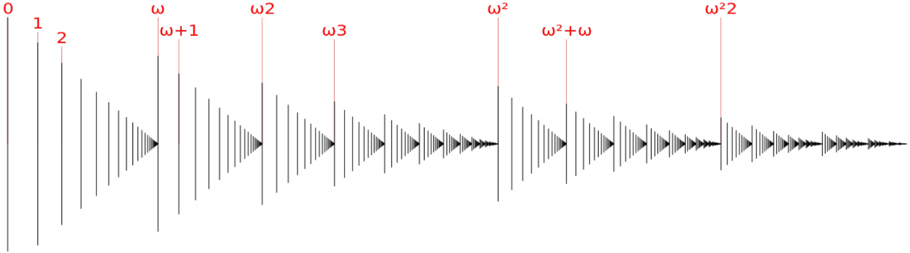

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First problems

We say that a function $f : N \to \omega_1$ is regressive if $f(\alpha) < \alpha$ for all $\alpha \in N$.
For $X, Y \subseteq \omega_1$, we write $X <_R Y$ if there exist $\gamma < \omega_1$ and an injective regressive $f : X \setminus \gamma \to Y$.

Questions

When a subset $N \subseteq \omega_1$ admits an injective regressive function to ω_1?
More generally, characterize when $X <_R Y$ for $X, Y \subseteq \omega_1$.

Fact

Every nonstationary set admits a 2-1 regressive function.

Proposition

The nonstationary set $N_0 = \bigcup\{(\delta, \delta+\omega] : \delta \in \omega_1 \text{ limit}\}$ does not admit an injective regressive function.

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Proof.

Assume $f : N_0 \to \omega_1$ is 1-1 regressive.

$$g(\alpha) := \begin{cases} f^{-1}(\alpha) & \alpha \in img(f) \\ 0 & \text{otherwise} \end{cases}$$

Let $\delta \in C_g \cap \text{Lim}$.
$\alpha < \delta \implies g(\alpha) < \delta$.
Then $\beta \geq \delta \implies f(\beta) \geq \delta$.
But then f maps $(\delta,\delta+\omega]$ into $[\delta,\delta+\omega)$.
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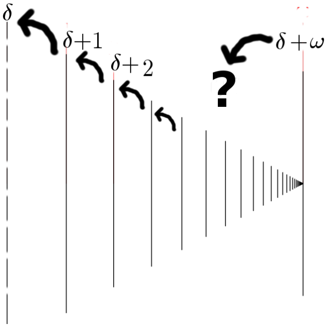

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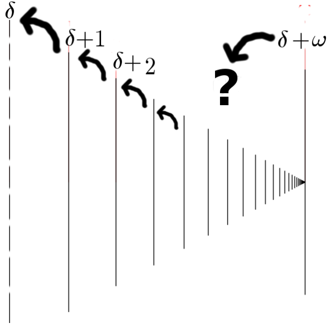

Lower bounds in $([\omega_1]^{\aleph_1}, <_R)$

Proposition

Every family of $\aleph_1$ subsets of ω_1 has a lower bound.

Proof.

Let $X_\alpha \subseteq \omega_1$ $(\alpha \in \omega_1)$.

1. Define $X = \{x_\alpha : \alpha < \omega_1\} \subseteq \omega_1$ by:

$$x_\alpha := \sup\{X_\beta(\alpha) + 1 : \beta \leq \alpha\}.$$

2. The map

$$x \longmapsto X_\beta(\min\{\alpha : x_\alpha = x\})$$

   is well defined from X to X_β.

3. Since $x_\alpha > X_\beta(\alpha)$ for all $\alpha \geq \beta$, it is regressive on $X \setminus \{x_\alpha : \alpha < \beta\}$.

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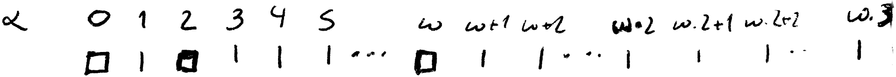
The combinatorics of $<_R$

Elements of X are indicated by a bar | (ordinals outside of X are denoted by a box □).



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α	0	1	2	3	4	5	…	ω	$\omega+1$	$\omega+2$	…	$\omega\cdot 2$	$\omega\cdot 2+1$	$\omega\cdot 2+2$	…	$\omega\cdot 3$
	□	|	□	|	|	|	…	□	|	|	…	|	|	|	…	|
A_α	1	1	2	2	2	2	…	2	2	2	…	1	1	1	…	0

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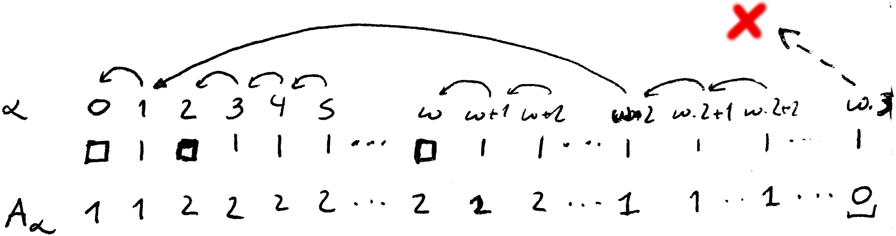

A-numbers

Given $X \subseteq \omega_1$, let $A_\alpha \in \mathbb{Z} \cup \{\pm\infty\}$ be defined by

$$A_0 := 1$$

$$A_{\alpha+1} := \begin{cases} A_\alpha & \alpha+1 \in X \\ A_\alpha + 1 & \alpha+1 \notin X \end{cases} \tag{1}$$

$$A_\gamma := \begin{cases} \liminf_{\alpha<\gamma}(A_\alpha - 1) & \gamma \in X \\ \left(\liminf_{\alpha<\gamma}(A_\alpha - 1)\right) + 1 & \gamma \notin X \end{cases} \qquad \gamma \text{ limit,} \tag{2}$$

Straightforward extension A^δ_α if the “origin” is δ instead of 0 (and $X \subseteq [\delta, \omega_1)$).

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Straightforward extension A^{δ}_{α} if the “origin” is δ instead of 0 (and $X \subseteq [\delta, \omega_1)$).

Characterization of $<_R$ for ω_1

We first characterize the non-$<_R$-maximal subsets of ω_1.

Theorem

Assume $X \subseteq \omega_1$. The following are equivalent:

1. *There exists a 1-1 regressive function $f : X \to \omega_1$;*
2. *There exists a club $C \subseteq \omega_1$ such that $C \cap X = \varnothing$ and for all $\delta \in C$, $A^\delta_\alpha > 0$ for $\alpha \in X$, or there exists $\beta \in (\delta, \delta^+)$ such that the former holds for $\alpha < \beta$ and $A^\delta_\beta = \omega$.*

For the characterization of $<_R$, the existence of an injective regressive map from X into Y really depends on the relative position of each of them in $Z := X \cup Y$.

$$A_0 := 1$$
$$A_{\alpha+1} := A_\alpha - \chi_X(Z(\alpha+1)) + \chi_Y(Z(\alpha+1))$$
$$A_\gamma := \liminf_{\alpha<\gamma}\big(A_\alpha - \chi_Y(Z(\alpha))\big) - \chi_X(Z(\gamma)) + \chi_Y(Z(\gamma))$$

Characterization of $<_R$ for ω_1

We first characterize the non-$<_R$-maximal subsets of ω_1.

Theorem

Assume $X \subseteq \omega_1$. The following are equivalent:

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$<\beta-1$ regressive maps

Let $\beta \leq \omega_1$.

Definition

$X <_R^\beta Y$: For some $\gamma < \omega_1$, $\exists f : X \setminus \gamma \to Y$ regressive such that

$$\forall y < \omega_1,\ \text{order-type}(f^{-1}(y)) < \beta.$$

Note:

$X <_R^2 Y$ iff $X <_R Y$
$X <_R^{\omega_1} Y$ iff there is a countable-to-one regressive $f : X \setminus \{0\} \to Y$.

Further results

Proposition

There exist $2^{\aleph_1}$ ($<_R^{\omega_1}$)-maximal nonstationary sets.

Proof.

1. Partition ω_1 into $\aleph_1$ stationary set $S_0, \ldots, S_{\omega_1}$.
2. Given $Y \subseteq \omega_1$, translate S_α by 1 for $\alpha \in Y$.
3. Take the union of the whole thing:

$$\bigcup\{S_\alpha + 1 : \alpha \in Y\} \cup \bigcup\{S_\alpha : \alpha \in (\omega_1 \setminus Y) \cup \{\omega_1\}\}.$$

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Future work

- We have characterizations of $<_R^\beta$ on ω_1 for $\beta \leq \omega + 1$. We plan to extend them for arbitrary $\beta < \omega_1$
- Apply Milner-Rado pigeonhole principles (e.g. the eponymous "paradox") to the case of $<_R^\beta$ on cardinals $\kappa > \omega_1$).

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Thank you!