Logics for Markov Decision Processes

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Joint work with P.R. D'Argenio and N. Wolovick

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A toy model

Labelled Transition Systems (LTS)

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Zig-zag morphism

A surjective $f : S \to S'$ such that for all $a \in L$ and every $s \in S$, $\mathrm{Pow}(f) \circ T_a = T'_a \circ f$.

lts02.jpg

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We say that s **simulates** t because s can perform every “sequence of actions” that t can.

Simulation and Bisimulation on LTS

Simulation

It is a relation R such that if $s_1 \, R \, t_1$ and $t_1 \xrightarrow{a} t_2$ then there is s_2 such that $s_1 \xrightarrow{a} s_2$ and $s_2 \, R \, t_2$. In that case we say that s_1 *simulates* s_2.

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Bisimulation

It is a **symmetric** simulation. We'll say that s_1 is *bisimilar* to t_1 if there exists a bisimulation R such that $s_1 \, R \, t_1$.

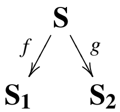
lts12.jpg

Note: Bisimulation is finer than "double simulation". That's to say, if s_1 is bisimilar to t_1, then s_1 simulates t_1 and t_1 simulates s_1, **but not conversely**.

Coalgebraic presentation of processes and bisimulation

One categorical counterpart of a relation is a *span* of morphisms

Bisimilarity (span)



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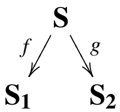
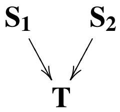


Behavioral equivalence **(cospan)**



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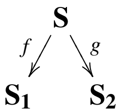

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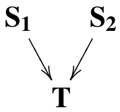

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Semipullbacks

A category *has semipullbacks* if every cospan can be completed to a commutative diagram with a span.

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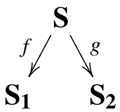

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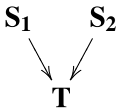

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Semipullbacks

A category *has semipullbacks* if every cospan can be completed to a commutative diagram with a span.

It is the Amalgamation Property in the opposite category.

Logics for Bisimulation

Hennessy-Milner Logic (HML)

$$\varphi \equiv \top \mid \neg\varphi \mid \bigwedge_i \varphi_i \mid \langle a \rangle \psi$$

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"t_1 can make an a-transition after which a c-transition is not possible".

$$t_1 \models \langle a \rangle \neg \langle c \rangle \top$$

$$s_1 \not\models \langle a \rangle \neg \langle c \rangle \top$$

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Logical Characterization of Bisimulation

Two states in a LTS are bisimilar iff they satisfy the same HML formulas.

Labelled Markov Processes (LMP) and Non Determinism

LMP (Desharnais et al.)

$\langle S, \mathcal{S}, L, t \rangle$ such that $t_a(s) \in \mathbf{P}(\mathcal{S})$ for each $s \in S$ and $a \in L$, where

- $\langle S, \mathcal{S} \rangle$ is a measurable space;
- $\mathbf{P}(\mathcal{S})$ is the space of (sub)probability measures over $\langle S, \mathcal{S} \rangle$;
- $t_a : S \to \mathbf{P}(\mathcal{S})$ is measurable.

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NLMP (D'Argenio and Wolovick)

$\langle S, \mathcal{S}, L, T\rangle$ such that $T_a(s) \subseteq \mathbf{P}(S)$ para each $s \in S$ y $a \in L$, where:

- $\langle S, \mathcal{S}\rangle$, $\mathbf{P}(S)$ as before;
- For each s, $T_a(s)$ is measurable. I.e., $T_a : S \to \mathbf{P}(\mathcal{S})$.
- $T_a : S \to \mathbf{P}(\mathcal{S})$ is a measurable map.

A pinch of Descriptive Set Theory: Analytic Spaces

Definition

An *analytic* topological space is the continuous image of a Borel set (v.g., of reals).

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Examples

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- The relation of isomorphism between countable structures.

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Unique Structure Theorem

If a sub-σ-algebra $\mathcal{S} \subseteq \mathbf{B}(A)$ is countably generated and separates points, then it is $\mathbf{B}(A)$.

Logics for bisimulation on LMP

HML_q (Larsen and Skou, Danos *et al.*)

$$\varphi \equiv \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle_q \varphi, \qquad q \in \mathbb{Q}$$

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Logical Characterization of Bisimulation for LMP (Danos *et al.*)

Two states in a LMP $\langle S, \mathcal{S}, L, t \rangle$ with $\langle S, \mathcal{S} \rangle$ *analytic* are bisimilar iff they satisfy the same HML_q formulas.

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Proof Strategy (D'Argenio, Celayes, PST)

This results holds for every process with an analytic state space and a logic $\mathcal{L}$ that satisfies: 1) $\mathcal{L}$ it contains $\top$ and $\wedge$; 2) for every $\varphi \in \mathcal{L}$, $[\![\varphi]\!]$ is measurable; 3) $\mathcal{L}$ is countable; and 4) $\mathcal{L}$ separates transitions "locally".

Logics for bisimulation on LMP

$\mathcal{L}_f$ (D'Argenio *et. al*)

$$\varphi \equiv \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle \{\varphi_i, p_i\}_{i=1}^{n}, \qquad p_i \in \mathbb{Q},\ n \in \mathbb{N}$$

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The proof strategy immediately gives

Logical Characterization of Bisimulation for image finite NLMP

Two states in a image finite NLMP $\langle S, \mathcal{S}, L, t \rangle$ with $\langle S, \mathcal{S} \rangle$ analytic are bisimilar iff they satisfy the same $\mathcal{L}_f$ formulas.

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Δ (D'Argenio *et. al*)

$$\varphi \equiv \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle \psi$$

$$\psi \equiv \bigvee_{i \in I} \psi_i \mid \neg \psi \mid [\varphi]_{\geq q}$$

Some counterexamples

Analiticity is necessary

The category of LMP over arbitrary measurable spaces does not have semipullbacks and HML_q does not characterize bisimilarity (PST, *Inf& Comp.* **209** 2011)

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At least image-countable is necessary

For NLMP over analytic spaces (D'Argenio, PST, Wolovick, *Math.Struct.Comp.Sci,* **22** 2009).

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Future Work

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- If possible, to extend the logical characterization to Radon spaces $\langle S, \mathcal{S} \rangle$ (i.e., $\mathcal{S} \subseteq$ universally measurable sets).

Thank You!

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